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Generation and Comparison of Globally Isotropic Space-Filling Truss Structures

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The purposes of this paper are to present a rationale for obtaining space-filling truss structures that behave like a globally isotropic continuum and to use continuum modeling to investigate their relative structural efficiencies (e.g., modulus-to-density, strength-to-density, and part-count-to-volume ratios). The trusses considered herein are generated by replication of a characteristic truss cell uniformly through space. The characteristic cells are categorized by one of a set of possible geometric symmetry groups derived using the techniques of crystallography. The implied elastic symmetry associated with each geometric symmetry group is identified to simplify the task of determining stiffness tailoring rules for guaranteeing global isotropy. Four truss geometries are analyzed to determine stiffness tailoring necessary for isotropy. All geometries exhibit equivalent isotropic Poisson's ratios of 1/4 and equivalent modulus-to-density ratios of 1/6 times the modulus-to-density ratio of the material used in their members. The truss configuration that has the lowest percent difference in member lengths is shown to have the lowest component part-count-to-volume ratios of all geometries considered when compared on a basis of equal stiffness, equal strength, and equal mass.

Nomenclature

A_n = cross-sectional area of members in nth group
$(A_{\text{proj}})_n$ = projected area per member for nth group of parallel members
C_{ijkl} = equivalent elastic constants (tensor form)
$(C'_{1111})_n$ = equivalent unidirectional stiffness for nth group of parallel members
c_{mn} = equivalent elastic constants (matrix form)
E = Young's modulus of truss material
E_{eq} = equivalent global Young's modulus of truss
I_n = cross-sectional moment of inertia of truss members in nth group
L = characteristic dimension of truss repeating cell
$L_{\max}$ = maximum member length in truss
$L_{\min}$ = minimum member length in truss
L_n = length of members in nth group
r_n = radius of gyration of members in nth group
T_{ij} = symmetry transformation tensor
α = length ratio of repeating truss cell in y direction
β = length ratio of repeating truss cell in z direction
$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$ = engineering shear strains
ΔL = percent difference in minimum and maximum member lengths
δ_n = ratio of cross-sectional area of members in nth group to that of first group
ϵ_{ij} = tensor strains
$\epsilon_x, \epsilon_y, \epsilon_z$ = engineering normal strains
ν_{eq} = equivalent Poisson's ratio of truss
ρ = density of truss material
ρ_{eq} = equivalent global density of truss
σ_{ult} = global compression strength of truss
σ_{ij}, τ_{ij} = normal and shear stresses

Introduction

FOR more than a decade, numerous studies have been conducted to determine the feasibility and structural characteristics of very large orbiting spacecraft incorporating lightweight truss structures. Due to the great size of these structures, and the large number of members comprising them, efficient techniques have been developed for static and dynamic analysis of trusses using an equivalent continuum analogy.[1] Although less versatile than general purpose structural analysis codes, the equivalent continuum analysis technique is attractive because it reduces the size of the analysis problem by relating the global mechanics of complex reticulated structures to the simpler, well-known behavior of an anisotropic continuum. This provides the analyst better insight into the structural behavior of large truss structures and aids in their preliminary design. Another significant attribute of continuum analysis that has seen very little application is the use of the equivalent continuum properties to study the relative structural efficiencies of different truss configurations.

The purposes of the present study are to present a rationale, based on truss-member stiffness tailoring, for obtaining truss configurations that behave like a globally isotropic continuum and to use continuum analysis to investigate their relative structural efficiencies (e.g., modulus-to-density, strength-to-density, and part-count-to-volume ratios). The trusses considered herein are generated by replication of a characteristic cell uniformly through space. This type of replication corresponds to an equivalent continuum that is homogeneous. In most cases, the repeating cells that generate homogeneous trusses inherently possess some geometric symmetry. The presence of geometric symmetry implies elastic symmetry that reduces the number of independent equivalent elastic constants character-

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izing the truss. Thus, most uniform space-filling trusses are specially anisotropic, and this information is used to greatly simplify the task of determining stiffness tailoring rules for guaranteeing global isotropy.

In this study, the techniques of crystallography are used to define the possible geometric symmetry groups associated with repeating cells that generate uniform trusses. In addition, the type of anisotropy associated with each geometric symmetry group is identified. Based on this information, stiffness tailoring rules for guaranteeing global isotropy are presented for four different truss configurations. Finally, the equivalent global modulus-to-density, strength-to-density, and part-count-to-volume ratios of these trusses are compared to determine the most efficient truss configuration. From this information, guidelines are presented for the design of high-efficiency globally isotropic trusses.

Geometric and Elastic Symmetry

The uniform truss structures considered herein are similar to crystalline lattices since they both can be generated by replication of a characteristic repeating cell that typically possesses geometric symmetry. This symmetry may include symmetry with respect to specific rotations about one or more axes and/or symmetry with respect to reflection about one or more planes. An important consequence of geometric symmetry is the implied elastic symmetry in the equivalent continuum constitutive equations. This implied elastic symmetry reduces the number of independent equivalent elastic constants characterizing the global behavior of the truss and thus simplifies the task of developing stiffness tailoring rules that guarantee global isotropy.

Rotational Symmetry Groups

Studies in crystallography[2,3] have shown that the rotational and reflection symmetries in reticulated or discrete structures are limited to a set of 32 possible combinations that are commonly called crystallographic symmetry groups. Love[4] determined that the elastic behavior of most crystallographic symmetry groups can be derived by considering only rotational symmetry. For brevity, the few cases in which reflection symmetry is important are not considered herein. By neglecting reflection symmetry, the 32 crystallographic symmetry groups reduce to 10 rotational symmetry groups, 8 of which are shown in Fig. 1. The remaining two rotational symmetry groups are specializations of groups *g* and *h* and are not shown in Fig. 1 because they have the same elastic behavior as groups *g* and *h*, respectively.

Each symmetry group in Fig. 1 is identified by a specific combination of axes about which there is rotational symmetry. The orientations of these axes are shown relative to a Cartesian coordinate system, and the order of rotational symmetry is given by one of four graphical symbols: a cusped oval, a triangle, a square, or a hexagon. These symmetry symbols are related to the order of symmetry in the accompanying key. The order of symmetry is defined as *n*-gonal where the rotation angle is $2\pi/n$ and *n* is either 2, 3, 4, or 6. Notice that in symmetry group h the trigonal symmetry axes lie along lines connecting the center of a cube with its corners. Thus structures of this symmetry group are often referred to as "cubic" structures.

Symmetry groups that possess only one axis of rotational symmetry are called monoaxial, and those that possess more than one are called multiaxial. The three rotational symmetry axes presented for each of the multiaxial groups are not the only symmetry axes possessed by those groups. A complete set can be generated by applying the symmetry operation of each axis to the others. For example, in symmetry group *d*, applying trigonal symmetry about the *z* axis identifies four additional digonal symmetry axes in the *x-y* plane separated by 60 deg.

Any truss structure that possesses axes of rotational symmetry can be categorized by one of the eight rotational symmetry groups in Fig. 1. This classification can be accomplished by identifying all rotational symmetry axes within the structure and then selecting a Cartesian coordinate system relative to these axes that matches one of the given symmetry groups. Once the symmetry group of the truss is identified, its inherent elastic behavior is determined using the methods presented in the next section.

Inherent Elastic Behavior

A uniform truss structure can be represented by a homogeneous equivalent anisotropic continuum characterized by 21 empirical elastic constants. These elastic constants, c_{mn} or C_{ijkl}, appear in the constitutive equations given in Eq. (1a) in matrix form and Eq. (1b) in tensor form. When the continuum possesses geometric symmetry, elastic symmetry is implied that reduces the number of independent elastic constants:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{1a}$$

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \tag{1b}$$

A truss structure that possesses geometric symmetry with respect to a linear orthogonal transformation T_{ij}, such as a

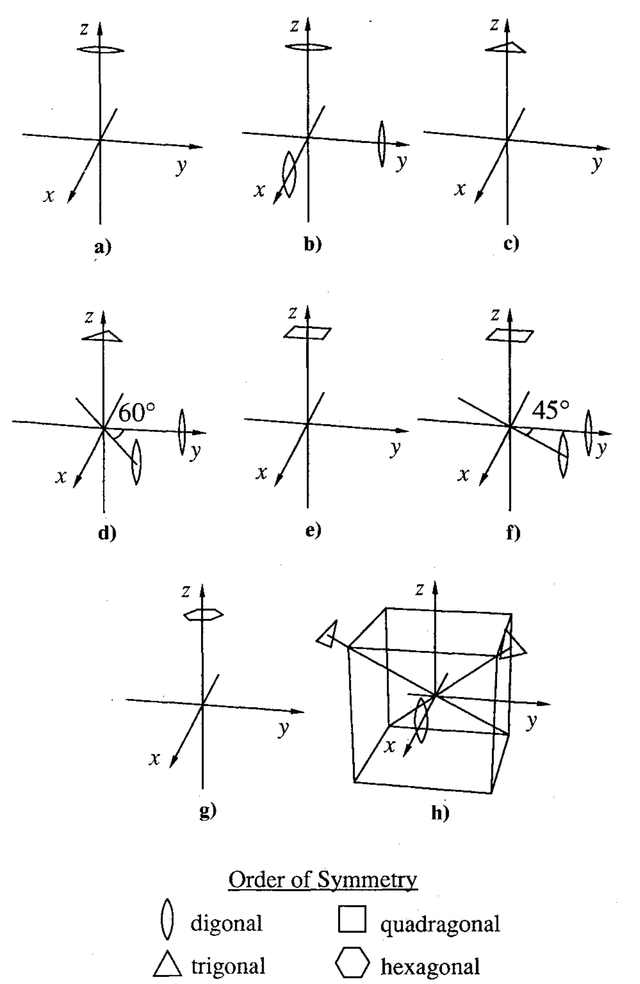

Fig. 1 Possible rotational symmetry groups for space-filling truss structures.

rotation or a reflection, also possesses symmetry in its equivalent continuum elastic constants. Therefore, the transformed constitutive tensor C'_{mnop}, given in Eq. (2a),[5] must be identical to the original tensor C_{ijkl}. Enforcing this requirement gives the set of conditions in Eq. (2b):

$$C_{ijkl} = C'_{mnop} T_{mi} T_{nj} T_{ok} T_{pl} \tag{2a}$$

$$C_{ijkl} = C_{mnop} T_{mi} T_{nj} T_{ok} T_{pl} \tag{2b}$$

The number of independent elastic constants associated with each symmetry group, presented in Fig. 1, is determined using Eq. (2b). A transformation tensor T_{ij} is determined for the specified rotation about each symmetry axis and substituted into Eq. (2b) to give 21 conditions on the elastic constants C_{ijkl}. Some of these conditions will be identically satisfied, whereas others can only be satisfied by the elimination or restriction of certain elastic constants. This process is repeated for all rotational symmetry axes in the given symmetry group, and the resulting reduced set of elastic constants defines the global elastic behavior of any truss structure that is a member of that symmetry group.

For example, the independent elastic constants characterizing trusses of symmetry group *a* are determined by enforcing elastic symmetry with respect to a 180-deg rotation symmetry about the *z* axis. The transformation matrix for this rotation can be written as

$$T_{ij} = \begin{cases} \delta_{ij}, & \text{for} \quad i = j = 3 \\ -\delta_{ij}, & \text{for} \quad i = j = 1 \text{ or } 2 \end{cases} \tag{3}$$

where δ_{ij} is the Kronecker delta. Substitution of Eq. (3) into the Eq. (2b) gives the following result:

$$C_{ijkl} = \begin{cases} C_{ijkl}, & \text{if an even number (or none) of the indices are 3} \\ -C_{ijkl}, & \text{if an odd number of the indices are 3} \end{cases} \tag{4}$$

Satisfying the later of the two conditions in Eq. (4) requires the following to be true (note that, due to symmetry in C_{ijkl}, many possible permutations of the subscripts have been omitted):

$$C_{1123} = C_{1113} = C_{2223} = C_{2213} = C_{3323} = C_{3313} = C_{2312} = C_{1312} = 0 \tag{5}$$

Employing the usual conversion from tensor to matrix form, the following equivalent conditions exist for the matrix elastic constants:

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0 \tag{6}$$

Similar calculations can be made for the remaining symmetry groups in Fig. 1. Without presenting the details, Table 1 shows the conditions on elastic constants as well as the number of independent elastic constants for each symmetry group.

As explained by Rosen and Shu,[6] and seen in Table 1, none of the permissible geometric symmetry groups possess sufficient symmetry to insure isotropic elastic behavior in uniform trusses. However, isotropy can be obtained by tailoring the relative stiffnesses of different truss members. The rationale behind this idea is presented in the next section.

Stiffness Tailoring for Isotropy

Changing the relative axial stiffness of different members in a truss changes the equivalent global elastic constants of the truss. This type of stiffness tailoring can be used to force certain trusses to behave isotropically. In the present study, the equivalent global elastic constants of a truss are calculated in terms of a set of geometric and/or stiffness parameters, and values are determined for these parameters that produce global isotropy. Parameters that are varied include dimensions of the truss repeating cell and relative axial stiffnesses of different truss members within the repeating cell.

Only dimensions and member stiffnesses that can be changed without violating the geometric symmetry of the repeating cell are varied. This restriction allows the cell to remain in the same geometric symmetry group and the conditions on its equivalent elastic constants given in Table 1 to remain valid. Members within the repeating cell that can be made coincident by applying rotational transformations from the geometric symmetry group of the cell must have the same stiffness and length. Therefore, the relative lengths and stiffnesses of members that cannot be made coincident by applying symmetry transformations are the only parameters that are varied.

For isotropy, the number of independent equivalent elastic constants characterizing a truss of a given symmetry group must be reduced to two from the number given in Table 1. Therefore, a truss can only be tailored to behave isotropically if the number of dimension and stiffness parameters that can be varied is two less than the number of independent elastic constants indicated in Table 1. For example, a truss of symmetry group h has three independent elastic constants, and it can be tailored to behave isotropically if it has at least one variable dimension or stiffness parameter.

Once a candidate truss for isotropic stiffness tailoring is selected, its equivalent elastic constants are calculated in terms of the variable parameters. The approach used in this study for calculating these elastic constants was developed by Nayfeh and Hefzy[7] and can be thought of as a three-dimensional generalization of classical laminated plate theory.[8] In this procedure, elastic constants for each group of parallel members are first determined relative to a local coordinate system with its x' axis lying in the direction of the members. Then these constants are transformed into a global coordinate system and summed with those from the other groups of parallel members.

Each group of parallel members is considered to be a unidirectional elastic continuum that has no transverse or shearing stiffnesses according to the basic definition of a truss member. Thus, each group of parallel members is characterized by only one nonzero equivalent stiffness that is in the local x' direction (along the direction of the members in the group). This equivalent unidirectional stiffness is determined from an area average of the member axial stiffness and is given in Eq. (7) for the nth group of members. The variables A_n and $(A_{\text{proj}})_n$ are defined as the cross-sectional area of each member and the projected area in space occupied by each member, respectively. The projected area is determined by passing a plane through the truss perpendicular to the group of members and calculating the portion of area in the plane attributed to one

Table 1 Elastic behavior of symmetry groups

Symmetry group[a]	Conditions	Number of constants
No symmetry	None	21
a	$c_{14}, c_{15}, c_{24}, c_{25}, c_{34}, c_{35}, c_{46}, c_{56} = 0$	13
b	Same as *a* along with $c_{16}, c_{26}, c_{36}, c_{45} = 0$	9
c	$c_{16}, c_{26}, c_{34}, c_{35}, c_{36}, c_{45} = 0$, $c_{11} = c_{22}$, $c_{44} = c_{55}$, $c_{13} = c_{23}$, $c_{14} = -c_{24} = c_{56}$, $c_{15} = -c_{25} = -c_{46}$, $c_{66} = (c_{11} - c_{12})/2$	7
d	Same as *c* along with $c_{15}, c_{25}, c_{46} = 0$	6
e	Same as *a* along with $c_{36}, c_{45} = 0$, $c_{11} = c_{22}$, $c_{44} = c_{55}$, $c_{13} = c_{23}$, $c_{16} = -c_{26}$	7
f	Same as *e* along with $c_{16}, c_{26} = 0$	6
g	Same as *c* along with c_{14}, c_{15}, c_{24}, $c_{25}, c_{46}, c_{56} = 0$	5
h	Same as *b* along with $c_{11} = c_{22} = c_{33}$, $c_{12} = c_{13} = c_{23}$, $c_{44} = c_{55} = c_{66}$	3

[a]See Fig. 1.

member (additional details on the calculation of the projected area are given in Ref. 7):

$$(C'_{1111})_n = \frac{EA_n}{(A_{\text{proj}})_n} \tag{7}$$

The equivalent continuum elastic constants for a truss are calculated by transforming the unidirectional constants for each of its groups of parallel members into a global coordinate frame using Eq. (2a) and summing the results as indicated by Eq. (8). The resulting equivalent elastic constants are explicit functions of the variable dimension and stiffness parameters selected for the truss. Therefore, forcing the equivalent elastic constants to become isotropic gives a set of equations that can be solved for the variable dimension and stiffness parameters:

$$C_{ijkl} = \sum_n (C'_{1111})_n (T_{1i}T_{1j}T_{1k}T_{1l})_n \tag{8}$$

The following sections present stiffness tailoring results for a variety of space-filling truss geometries.

Warren Truss

The repeating cell for a commonly used truss geometry, often referred to as the Warren truss, is shown in Fig. 2. A larger cell is generated from this repeating cell by replicating it with rotations about the z axis in increments of 90 deg (see Fig. 3) and replicating the resulting four cells with a 180-deg rotation about either the x or y axes. This larger cell is the basic unit from which a space-filling lattice can be generated by simple translational replication in the x, y, and z directions and is referred to as the basic translational replication unit.

The Warren truss repeating cell shown in Fig. 2 is of symmetry group h (see Fig. 1), and the resulting space-filling truss lattice is also of symmetry group h. From Table 1, it is seen that, in general, trusses of this symmetry group have three independent elastic constants: c_{11}, c_{12}, and c_{66}. In addition, the Warren truss has two different length members whose relative axial stiffnesses can be varied without altering its geometric symmetry. A value can be determined for the ratio of these stiffnesses that eliminates one independent elastic constant and thus makes the Warren truss behave isotropically.

The Warren truss is composed of nine different member groups. Three groups correspond to the members forming the edges of the cube, and six groups correspond to the diagonal members in the faces of the cube. The equivalent uniaxial stiffnesses of the three groups of edge members are the same, and the value is given by Eq. (9). Similarly, the equivalent uniaxial stiffnesses of the six groups of diagonals are the same, and the value is given by Eq. (10):

$$(C'_{1111})_1 = \frac{EA_1}{L^2} \tag{9}$$

$$(C'_{1111})_2 = \frac{EA_2}{\sqrt{2}L^2} \tag{10}$$

The transformation tensors T_{ij} for each group [see Eq. (8)] are orthogonal rotational transformation matrices that align the global x axis with the longitudinal direction of the members in the respective groups (local x' axis). Substituting the uniaxial stiffnesses from Eqs. (9) and (10) and the appropriate transformation tensors into Eq. (8) and simplifying gives the following stiffness matrix of the equivalent continuum where $\delta_2 \equiv A_2/A_1$.

$$[c_{mn}] = \frac{EA_1}{L^2}\begin{bmatrix} \left(1+\frac{\delta_2}{\sqrt{2}}\right) & \frac{\delta_2}{2\sqrt{2}} & \frac{\delta_2}{2\sqrt{2}} & 0 & 0 & 0 \\ \frac{\delta_2}{2\sqrt{2}} & \left(1+\frac{\delta_2}{\sqrt{2}}\right) & \frac{\delta_2}{2\sqrt{2}} & 0 & 0 & 0 \\ \frac{\delta_2}{2\sqrt{2}} & \frac{\delta_2}{2\sqrt{2}} & \left(1+\frac{\delta_2}{\sqrt{2}}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta_2}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\delta_2}{2\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\delta_2}{2\sqrt{2}} \end{bmatrix} \tag{11}$$

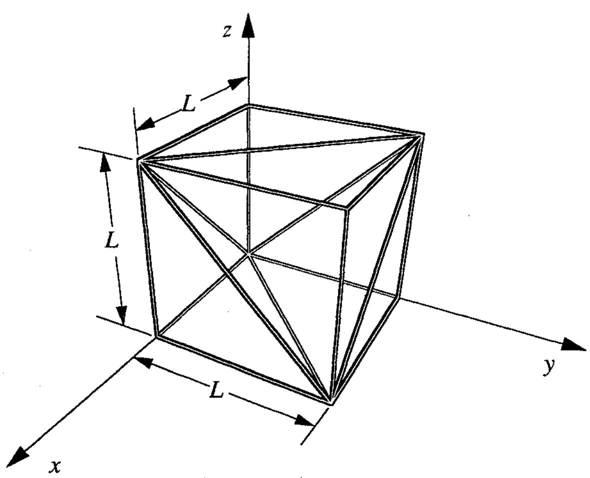

Fig. 2 Repeating cell for Warren truss.

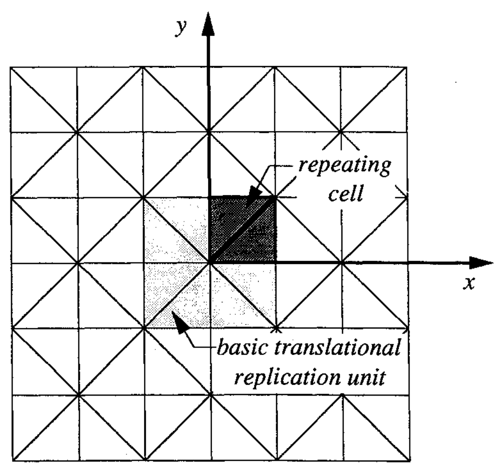

Fig. 3 Warren truss projected in the *x-y* plane.

It should be noted that the stiffness matrix in Eq. (11) is indeed of the form presented in Table 1 for structures of symmetry group h. The only condition remaining to be satisfied for isotropy is that $c_{66} = (c_{11} - c_{12})/2$. Enforcing this condition gives

$$\delta_2 = 2\sqrt{2}, \quad c_{11} = \frac{3EA_1}{L^2}, \quad c_{12} = \frac{EA_1}{L^2} \tag{12}$$

The equivalent isotropic Poisson's ratio and Young's modulus of the Warren truss are given in Eqs. (13) and (14), respectively. A Poisson's ratio of 1/4 was also determined by Nayfeh and Hefzy[7] for an isotropic three-dimensional truss. It will be shown that all truss geometries considered herein will also have equivalent Poisson's ratios of 1/4:

$$\nu_{eq} = c_{12}/(c_{11} + c_{11}) = 1/4 \tag{13}$$

$$E_{eq} = c_{11}\frac{(1+\nu_{eq})(1-2\nu_{eq})}{(1-\nu_{eq})} = \frac{5EA_1}{2L^2} \tag{14}$$

An equivalent density is also calculated for the Warren truss by adding the total member mass in the repeating cell and dividing by the volume of the repeating cell. This approach neglects any lumped mass associated with truss nodes. Without showing the details, the result is

$$\rho_{eq} = \frac{15\rho A_1}{L^2} \tag{15}$$

Finally, the modulus-to-density ratio for the Warren truss is determined in Eq. (16) for evaluation of the stiffness efficiency of the structure. Equation (16) indicates that this equivalent efficiency is dependent only on the efficiency of the material out of which the members are made:

$$\frac{E_{eq}}{\rho_{eq}} = \frac{1}{6}\left(\frac{E}{\rho}\right) \tag{16}$$

Body-Centered Cubic Truss

Another truss geometry of symmetry group h is based on a node arrangement identical to the body-centered cubic atomic lattice structure.[9] A repeating cell from this truss is shown in Fig. 4. The node locations of the cell correspond to the corners and the center of the cube. Space is filled by simply replicating this cell uniformly in the x, y, and z coordinate directions. Thus, for the body-centered cubic truss, the basic translational replication unit is identical to the repeating cell. It is important to note that the repeating cell shown in Fig. 4 is not a kinematically stable truss. It has no torsional stiffness because all diagonals intersect at one point. Similarly, a truss beam built by replicating this cell in one direction is also kinematically unstable in torsion. However, replication in two or three orthogonal directions results in a truss that is kinematically stable.

The body-centered cubic truss is composed of seven different member groups. Three groups correspond to the members forming the edges of the cube, and four groups correspond to the diagonals connecting the center of the cube to its corners. The equivalent uniaxial stiffness of the three groups of edge members is the same as that of the Warren truss [see Eq. (9)]. The equivalent uniaxial stiffness of the diagonal member groups is given in Eq. (17). Substituting these uniaxial stiffnesses and the appropriate transformation tensors into Eq. (8) and simplifying gives a stiffness matrix of the same form as Eq. (11) where c_{11}, c_{12}, and c_{66} are given in Eq. (18):

$$(C'_{1111})_2 = \frac{2EA_2}{\sqrt{3}L^2} \tag{17}$$

$$c_{11} = \frac{EA_1}{L^2}\left(1 + \frac{4\delta_2}{3\sqrt{3}}\right), \qquad c_{12} = c_{66} = \frac{EA_1}{L^2}\left(\frac{4\delta_2}{3\sqrt{3}}\right) \tag{18}$$

Enforcing $c_{66} = (c_{11} - c_{12})/2$ for isotropy gives

$$\delta_2 = \frac{3\sqrt{3}}{8}, \quad c_{11} = \frac{3EA_1}{2L^2}, \quad c_{12} = \frac{EA_1}{2L^2} \tag{19}$$

Equivalent isotropic Poisson's ratio, Young's modulus, and density are calculated as described before, and their values are given in Eq. (20). Notice that the equivalent modulus-to-density ratio of the isotropic body-centered cubic truss is identical to the result detrmined for the isotropic Warren truss [see Eq. (16)]:

$$\nu_{eq} = 1/4, \quad E_{eq} = \frac{5EA_1}{4L^2}, \quad \rho_{eq} = \frac{15\rho A_1}{2L^2} \tag{20}$$

Orthogrid Truss

The orthogrid truss shown in Fig. 5 has the same member arrangement as the Warren truss, the only difference being that the repeating cell and the basic translational replication unit for the orthogrid truss are rectangular parallelepipeds rather than cubes. Thus, for $\alpha = \beta = 1$ the orthogrid truss is identical to the Warren truss. The orthogrid truss is a member of symmetry group b, which in general has nine independent elastic constants (see Table 1) and thus is equivalent to an orthotropic solid. The orthogrid truss possesses six different length members (i.e., five variable member area ratios) and two variable dimensions defined by α and β. Therefore, a sufficient number of independent parameters exist to tailor the orthogrid truss to behave isotropically.

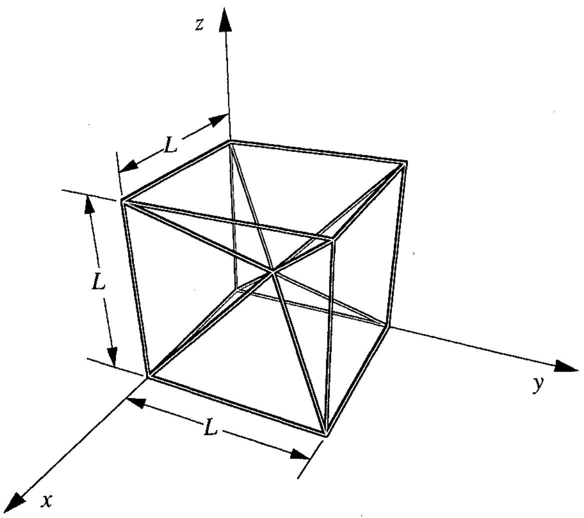

Fig. 4 Repeating cell for body-centered cubic truss.

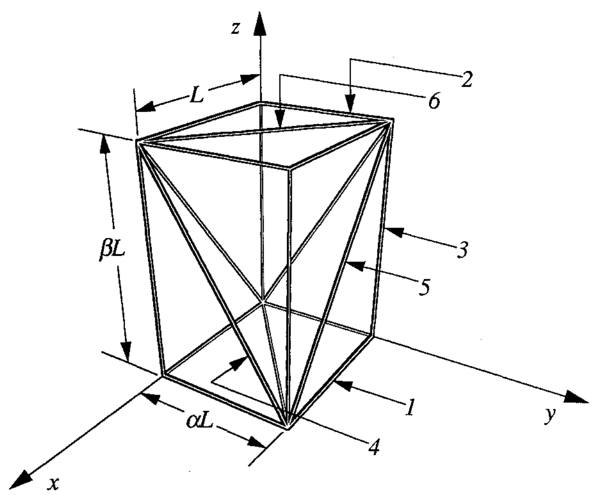

Fig. 5 Repeating cell for orthogrid truss.

The orthogrid truss has nine different member groups characterized by six different uniaxial stiffnesses. The groups lying in the x, y, and z directions are referred to as groups 1, 2, and 3, respectively, and those lying in the y-z, x-z, and x-y planes are referred to as groups 4, 5, and 6, respectively. The uniaxial stiffnesses for these groups are given in Eq. (21):

$$(C'_{1111})_1 = \frac{EA_1}{\alpha\beta L^2}, \quad (C'_{1111})_2 = \frac{EA_2}{\beta L^2}, \quad (C'_{1111})_3 = \frac{EA_3}{\alpha L^2}$$

$$(C'_{1111})_4 = \frac{(\alpha^2+\beta^2)^{1/2} EA_4}{2\alpha\beta L^2}, \quad (C'_{1111})_5 = \frac{(1+\beta^2)^{1/2} EA_5}{2\alpha\beta L^2}$$

$$(C'_{1111})_6 = \frac{(1+\alpha^2)^{1/2} EA_6}{2\alpha\beta L^2} \tag{21}$$

Substituting these uniaxial stiffnesses and the appropriate transformation tensors into Eq. (8) and simplifying gives the following values for the nonzero equivalent stiffness coefficients:

$$c_{11} = \frac{EA_1}{\alpha\beta L^2}\left[1 + \frac{\delta_5}{(1+\beta^2)^{3/2}} + \frac{\delta_6}{(1+\alpha^2)^{3/2}}\right] \tag{22}$$

$$c_{22} = \frac{EA_1}{\alpha\beta L^2}\left[\alpha\delta_2 + \frac{\alpha^4\delta_4}{(\alpha^2+\beta^2)^{3/2}} + \frac{\alpha^4\delta_6}{(1+\alpha^2)^{3/2}}\right] \tag{23}$$

$$c_{33} = \frac{EA_1}{\alpha\beta L^2}\left[\beta\delta_3 + \frac{\beta^4\delta_4}{(\alpha^2+\beta^2)^{3/2}} + \frac{\beta^4\delta_5}{(1+\beta^2)^{3/2}}\right] \tag{24}$$

$$c_{12} = c_{66} = \frac{EA_1}{\alpha\beta L^2}\left[\frac{\alpha^2\delta_6}{(1+\alpha^2)^{3/2}}\right] \tag{25}$$

$$c_{13} = c_{55} = \frac{EA_1}{\alpha\beta L^2}\left[\frac{\beta^2\delta_5}{(1+\beta^2)^{3/2}}\right] \tag{26}$$

$$c_{23} = c_{44} = \frac{EA_1}{\alpha\beta L^2}\left[\frac{\alpha^2\beta^2\delta_4}{(\alpha^2+\beta^2)^{3/2}}\right] \tag{27}$$

where $\delta_n \equiv A_n/A_1$ for $n = 2, 3, 4, 5$, and 6.

The orthogrid truss can be made to behave isotropically by satisfying the conditions stated in Eq. (28). Substituting the expressions in Eqs. (22–27) into Eq. (28) gives the results in Eqs. (29–33) for the required member area ratios:

$$c_{11} = c_{22} = c_{33}, \qquad c_{12} = c_{13} = c_{23}$$

$$c_{44} = c_{55} = c_{66} = (c_{11} - c_{12})/2 \tag{28}$$

$$\delta_2 = \frac{\alpha(3\beta^2 - \alpha^2 - \alpha^2\beta^2)}{3\alpha^2\beta^2 - \alpha^2 - \beta^2} \tag{29}$$

$$\delta_3 = \frac{\beta(3\alpha^2 - \beta^2 - \alpha^2\beta^2)}{3\alpha^2\beta^2 - \alpha^2 - \beta^2} \tag{30}$$

$$\delta_4 = \frac{(\alpha^2+\beta^2)^{3/2}}{3\alpha^2\beta^2 - \alpha^2 - \beta^2} \tag{31}$$

$$\delta_5 = \frac{\alpha^2(1+\beta^2)^{3/2}}{3\alpha^2\beta^2 - \alpha^2 - \beta^2} \tag{32}$$

$$\delta_6 = \frac{\beta^2(1+\alpha^2)^{3/2}}{3\alpha^2\beta^2 - \alpha^2 - \beta^2} \tag{33}$$

The equivalent Poisson's ratio, Young's modulus, and density of the orthogrid truss are found to be

$$\nu_{eq} = 1/4, \qquad E_{eq} = \frac{5EA_1}{2L^2}\left[\frac{\alpha\beta}{3\alpha^2\beta^2 - \alpha^2 - \beta^2}\right] \tag{34}$$

$$\rho_{eq} = \frac{15\rho A_1}{L^2}\left[\frac{\alpha\beta}{3\alpha^2\beta^2 - \alpha^2 - \beta^2}\right] \tag{35}$$

As for the Warren and body-centered cubic trusses, the equivalent isotropic Poisson's ratio for the orthogrid truss is exactly 1/4, and the equivalent modulus-to-density ratio is $E/6\rho$. Also, as expected, the values for the isotropic member area ratios, the equivalent Young's modulus, and the equivalent density are identical to those of the Warren truss if α and β are equal to 1.0.

The orthogrid truss was selected for study because it has just enough parameters (five member area ratios and two variable lengths) to enforce isotropy by eliminating seven of the nine independent elastic constants. However, some of the original elastic constants are not independent [see Eqs. (25–27)]. Therefore, the system of conditions is actually overdetermined, and isotropic solutions exist for ranges of α and β instead of only discrete values, as evidenced by the solutions given in Eqs. (29–33). The permissible ranges for α and β are determined by requiring that all area ratios δ_n be positive valued.

Because of the square root in their numerators, δ_4, δ_5, and δ_6 have positive and negative (or zero) values for any pair of α and β. Requiring δ_2 and δ_3 to be positive forces their numerators and denominators both to be either positive or negative. There are no values for α and β that cause all numerators and denominators to be negative; however, a range of values exist for which all terms are positive. This range of values is plotted as the lightly shaded region in Fig. 6. The bounding curves in this plot correspond to either one of the numerators or the denominator equaling zero, and the direction from the boundary in which the term is positive is indicated with arrows.

It is desirable to further restrict the values of δ_n to be within a fixed percentage of each other and thus avoid designs that have members with greatly different cross-sectional areas. Requiring that the maximum difference in member areas is a factor of five forces all δ_n and ratios of any two δ_n to be no greater than five or less than one fifth. Enforcing these constraints reduces the permissible range for α and β to the heavily shaded region shown in Fig. 6. This plot shows that it is only practical to consider values for α and β between about 0.85 and 1.15. Therefore, although the orthogrid truss can be tailored to behave isotropically, it is only practical if the lengths of the sides of its repeating cell differ by less than about 15%.

Isogrid Truss

The last truss to be considered in the present study is called the isogrid truss, and its repeating cell is shown in Fig. 7. A basic translational replication unit for the isogrid truss is generated by replicating the repeating cell about the z axis in rotations of 60 deg as shown in Fig. 8. A space-filling lattice is

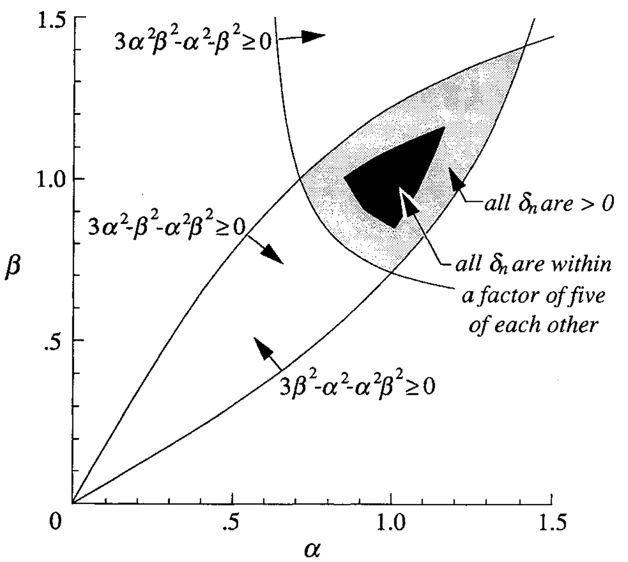

Fig. 6 Permissible ranges of lengths for orthogrid truss.

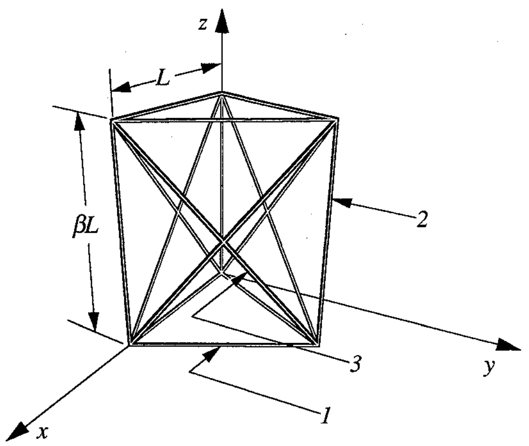

Fig. 7 Repeating cell for isogrid truss.

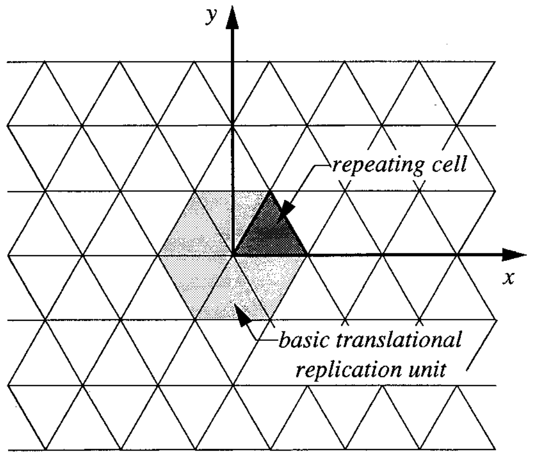

Fig. 8 Isogrid truss projected in x-y plane.

generated by simple translational replication in the y and z directions and along the lines $y = \pm x/\sqrt{3}$. This geometry is called an isogrid truss because the resulting network of members forms an isogrid in the x-y plane.

The repeating cell shown in Fig. 7 is of symmetry group d (see Fig. 1). However, due to the rotational replication, the basic translational replication unit and the resulting space-filling truss lattice are of symmetry group g. In general, truss geometries of symmetry group g have five independent elastic constants (see Table 1). Tailoring the relative axial stiffnesses of the three different length members and allowing the height of the repeating cell in the z direction to be a variable gives three parameters that can be tailored to reduce the five independent elastic constants to the two necessary for isotropy.

The isogrid truss has nine different member groups characterized by three different uniaxial stiffnesses. The equivalent uniaxial stiffnesses of the three groups of parallel members lying in the x-y plane are the same and the value is given in Eq. (36). The equivalent uniaxial stiffness for the members lying in the z direction is given in Eq. (37). The equivalent uniaxial stiffness for the remaining groups of parallel diagonal members is given in Eq. (38):

$$(C'_{1111})_1 = \frac{2EA_1}{3^{1/2}\beta L^2} \tag{36}$$

$$(C'_{1111})_2 = \frac{2EA_2}{3^{1/2}L^2} \tag{37}$$

$$(C'_{1111})_3 = \frac{2(1+\beta^2)^{1/2}EA_3}{3^{1/2}\beta L^2} \tag{38}$$

Substituting these stiffnesses and the appropriate transformation tensors into Eq. (8) and simplifying gives the following results for the nonzero equivalent elastic constants:

$$c_{11} = c_{22} = \frac{3^{3/2}EA_1}{4\beta L^2}\left[1 + \frac{2\delta_3}{(1+\beta^2)^{3/2}}\right] \tag{39}$$

$$c_{33} = \frac{3^{3/2}EA_1}{4\beta L^2}\left[\frac{8\beta\delta_2}{9} + \frac{16\beta^4\delta_3}{3(1+\beta^2)^{3/2}}\right] \tag{40}$$

$$c_{12} = c_{66} = \frac{3^{3/2}EA_1}{4\beta L^2}\left[\frac{1}{3} + \frac{2\delta_3}{3(1+\beta^2)^{3/2}}\right] \tag{41}$$

$$c_{13} = c_{23} = c_{44} = c_{55} = \frac{3^{3/2}EA_1}{4\beta L^2}\left[\frac{8\beta^2\delta_3}{3(1+\beta^2)^{3/2}}\right] \tag{42}$$

Enforcing the isotropy conditions given in Eq. (28) gives the following values for the member cross-sectional area ratios:

$$\delta_2 = \frac{3\beta(3-2\beta^2)}{2(4\beta^2-1)} \tag{43}$$

$$\delta_3 = \frac{(1+\beta^2)^{3/2}}{2(4\beta^2-1)} \tag{44}$$

The equivalent Poisson's ratio, Young's modulus, and density of the isogrid truss are found to be

$$\nu_{eq} = 1/4, \quad E_{eq} = \frac{15\beta EA_1}{12^{1/2}(4\beta^2-1)L^2} \tag{45}$$

$$\rho_{eq} = \frac{45\beta\rho A_1}{3^{1/2}(4\beta^2-1)L^2} \tag{46}$$

As before, the equivalent isotropic Poisson's ratio for the isogrid truss is 1/4, and the equivalent modulus-to-density ratio is $E/6\rho$. Also, similar to the orthogrid truss, not all five of the original elastic constants are independent [see Eq. (42)], and isotropic solutions exist for a range of β instead of only one discrete value. Once again, the practical range for β is determined by requiring that both δ_2 and δ_3 are between one fifth and five and that the ratio of these parameters also be between these limits. The minimum range for β is determined by forcing δ_2 to satisfy these restrictions. This range is between about 0.6 and 1.1 as shown in Fig. 9. Therefore, it is only practical to tailor the isogrid truss to behave isotropically, if its depth in the z direction is between 60 and 110% of the length of one of its members in the x-y plane.

All four trusses considered exhibit the same equivalent Poisson's ratio and equivalent modulus-to-density ratio. Thus, there is no way to discriminate between the geometries on the basis of stiffness alone. However, a comparison can be made by considering equivalent compressive strength and component part count.

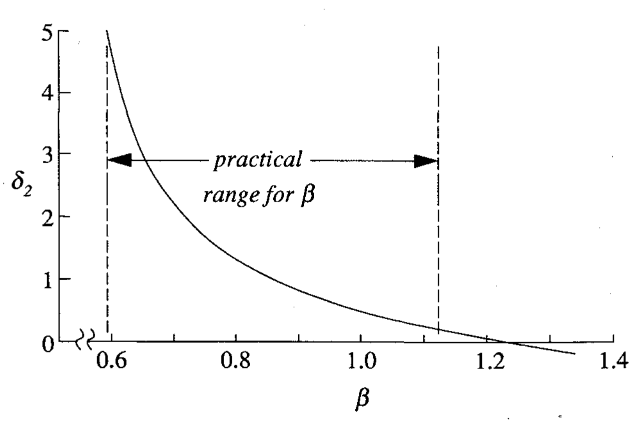

Fig. 9 Permissible range of length for isogrid truss.

Compressive Strength of Trusses

For the purpose of comparison, the strength of a truss is defined herein as the minimum global uniaxial compression stress it sustains before any of its members buckle elastically.[10] For an isotropic truss, the compression strain due to this globally applied stress is independent of orientation and is equal to

$$\epsilon = \frac{\sigma}{E_{eq}} \tag{47}$$

For any group of parallel members in a truss, the critical direction for application of the global compression load is the direction in which the members lie. Application of the stress along the direction of the members maximizes the end shortening of the members and the compressive load in the members. Therefore, the critical global compression stress is determined by equating the global strain in Eq. (47) to the Euler buckling strain of the members lying in the direction of the load application. Thus, the critical strain in the nth group of members is

$$\epsilon_n = \frac{\pi^2 EI_n/L_n^2}{EA_n} = \frac{\pi^2 I_n}{L_n^2 A_n} = \frac{\pi^2 r_n^2}{L_n^2} = \frac{\sigma}{E_{eq}} \tag{48}$$

where I_n, L_n, and r_n are the cross-sectional moment of inertia, length, and radius of gyration, respectively, of the members in group n.

Equation (48) indicates that the buckling strain in a member is dependent only on its slenderness ratio (L_n/r_n). If all members are thin-walled cylinders having the same diameter, their radii of gyration will be approximately equal (i.e., $r_n = r$) and therefore the member groups having the greatest length will have the lowest buckling strain. Thus, the compression strength of the truss is simply determined by setting the strain in Eq. (48) equal to the buckling strain of the longest members and solving for the stress. The result is given by

$$\sigma_{ult} = \pi^2 E_{eq} \frac{r^2}{L_{\max}^2} \tag{49}$$

Equations (48) and (49) lead to two observations regarding trusses having all members of the same diameter. First, the allowable compression stress varies with orientation within a truss, and although these trusses are isotropic in stiffness, they are not isotropic in strength. Second, to minimize variations in the allowable compression stress, it is necessary to minimize the differences in member lengths, and thus the most efficient truss from a strength standpoint is one that has the smallest differences in member lengths.

The second observation is used to determine the most efficient member length ratios, α and β, for the orthogrid and isogrid trusses discussed earlier. Without difficulty it can be shown that the difference in the minimum and maximum member lengths for both of these trusses is minimized if both length ratios are equal to 1.0. This indicates that the Warren truss with a cubic repeating cell is actually the most efficient orthogrid truss.

Finally, two different trusses having the same equivalent stiffnesses and member diameters can be designed to have the same compression strengths by simply selecting the same length for the longest members in both trusses. This requirement determines the relative size of the repeating cells of two different trusses and presents the opportunity for discriminating between trusses based on component part count.

Component Part Count of Trusses

Component part count is simply the number of members and joints within a truss. For a given truss geometry, a part-count density can be determined by summing the components within the repeating cell and dividing by the volume of the cell. This part-count density can then be readily used for comparison with other truss geometries. From an operations and cost standpoint, it is typically desirable to minimize the component part count of the truss. Therefore, the most efficient truss is the one that has the lowest part-count density for a given compression strength and stiffness.

A summary of the part-count calculations for each of the trusses considered is given in Table 2 (note that the orthogrid truss is omitted because in its most efficient form it is identical to the Warren truss). The characteristic length and repeating cell volume are determined by setting the maximum member length in each truss equal to 1.0. The total number of nodes and members in each repeating cell are determined by summing each component divided by the number of repeating cells that share it (e.g., a node shared by eight repeating cells is considered 1/8 of a node). Finally, the node and member densities are determined by dividing the number of nodes and members by the volume of the repeating cell. Also included in Table 2 is the percent difference in the minimum length members and the maximum length members of each truss as determined by

$$\Delta L = \frac{L_{\max} - L_{\min}}{L_{\max}} \times 100 \tag{50}$$

The body-centered cubic truss is shown to have the lowest node and member densities of the geometries considered. This geometry also has the lowest difference in minimum and maximum member lengths. Therefore, the body-centered cubic truss is the most efficient of the isotropic trusses considered in the present study.

Concluding Remarks

A rationale has been presented for obtaining and comparing the efficiencies of space-filling truss structures that behave globally like an isotropic continuum. A finite set of possible geometric symmetry groups that can be possessed by uniform trusses was presented, and the implied elastic symmetry associated with each geometric symmetry group was identified to simplify the task of determining stiffness tailoring rules necessary for isotropy. It was shown that, by varying dimensions and member area ratios of a repeating cell, the equivalent global elastic constants of a truss can be varied. Furthermore, if the truss possess enough variable dimensions or member area ratios to provide a determinate set of stiffness tailoring conditions, the number of independent global elastic constants can be reduced to the two necessary for isotropy.

Four truss geometries were analyzed to determine stiffness tailoring necessary for isotropy. All geometries exhibited equivalent isotropic Poisson's ratios of 1/4 and equivalent modulus-to-density ratios of $E/6\rho$. It was determined that the compression strength of isotropic trusses having the same equivalent stiffness and using members with the same radii of gyration is determined by the maximum member length.

Table 2 Summary of part-count calculations

Truss geometry	L	Volume of cell	Number of nodes/cell	Number of members/cell	Node density	Member density	ΔL, %
Warren	0.707	0.354	1	6	2.828	16.97	29.3
Body-centered cubic	1.0	1.0	2	11	2.0	11.0	13.4
Isogrid	0.707	0.153	0.5	5	3.266	32.66	29.3

Therefore, by setting the maximum member lengths equal, the component part-count densities of the different truss geometries were compared on a basis of equal stiffness, equal strength, and equal mass. The body-centered cubic truss was shown to have approximately 30% lower component part-count densities than the Warren truss and approximately 45–55% lower part-count densities than the isogrid truss. Furthermore, the body-centered cubic truss has the lowest percent difference in minimum and maximum member lengths, thus the minimum variation in compression strength with orientation. For these reasons, the body-centered cubic truss is considered the most efficient of the geometries studied.

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